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A Stable Recurrence Relation for Trigonometric B-Splines

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In this paper we give results that lead to stable algorithms for computing with trigonometric splines. In particular we show that certain trigonometric *B*-splines satisfy a recurrence relation similar to the one for polynomial splines. We also show how these trigonometric *B*-splines can be differentiated, and a trigonometric version of Marsden's identity is given. The results are obtained by studying certain trigonometric divided differences.

1. INTRODUCTION

Schoenberg [5] studied trigonometric spline functions, which he defined as piecewise trigonometric polynomials of the form

$$a_0 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx). \tag{1.1}$$

It was shown, for example, that any trigonometric spline could be expressed as a linear combination of certain trigonometric *B*-splines. The latter are defined via certain divided differences, and as in the polynomial case these basis functions have local support.

Trigonometric splines have been one of the sources that have motivated the study of the more general Chebyshevian splines (see, for example, [6] and references given therein). However, in most cases it is assumed that one is dealing with a *complete* Chebyshev system $\varphi_1, \varphi_2, \ldots, \varphi_m$; i.e., $\varphi_1, \ldots, \varphi_j$ is a Chebyshev system for $j = 1, 2, \ldots, m$. Note that this property does not hold on $[0, 2\pi)$ for the system $1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots, \cos nx, \sin nx$.

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Thus many of the general results do not apply directly to the trigonometric case. However, most results in this paper are derived by a simple transformation from the system 1, $e^{ix}, \ldots, e^{i(m-1)x}$.

Also, the general study has not led to an analog of the stable recurrence relation (see de Boor [1], Cox [2])

$$Q_{j,m}(x) = \frac{(x - x_j)\, Q_{j,m-1}(x) + (x_{j+m} - x)\, Q_{j+1,m-1}(x)}{x_{j+m} - x_j}, \tag{1.2}$$

where $Q_{j,m}$ are polynomial B-splines of order m.

The main purpose of this paper is to give results that lead to stable algorithms for computing with trigonometric splines. We define a trigonometric spline of order m to be a piecewise function of the form (1.1) if $m = 2n + 1$ ($n \geqslant 0$ integer), and to be a piecewise function of the form

$$\sum_{k=1}^{n} (a_k \cos(k - \tfrac{1}{2})x + b_k \sin(k - \tfrac{1}{2})x) \tag{1.3}$$

if $m = 2n$ ($n \geqslant 1$ integer).

In Section 2 we study a trigonometric analog of divided differences. If m is odd these linear functionals have the same null space as the divided differences introduced by Schoenberg in [5]. However, in order to derive a recurrence relation similar to (1.2) for trigonometric B-splines, we have found it convenient to use another scaling. The trigonometric divided differences will be used in Section 3 to define our trigonometric B-splines $T_{j,m}$ or order m. We show that the functions $T_{j,m}$ satisfy the relation

$$T_{j,m}(x) = \frac{\left(\sin \frac{x - x_j}{2}\right) T_{j,m-1}(x) + \left(\sin \frac{x_{j+m} - x}{2}\right) T_{j+1,m-1}(x)}{\sin \frac{x_{j+m} - x_j}{2}}. \tag{1.4}$$

This analog of (1.2) gives a stable algorithm for calculating $T_{j,m}(x)$.

We also obtain certain formulas for derivatives of $T_{j,m}$, and an integral representation of trigonometric divided differences with the function $T_{j,m}$ as a kernel. Finally, a trigonometric analog of Marsden's identity is given.

2. Trigonometric Divided Differences

We start with some notation. For any integer $m \geqslant 1$ let S_m and $\tilde{S}_m$ be the m-dimensional comples linear spaces of functions on $\mathbb{R}$ generated by

$$\left\{\exp\left(-i\frac{m-1}{2}x\right), \exp\left(-i\left(\frac{m-1}{2} - 1\right)x\right), \ldots, \exp\left(i\frac{m-1}{2}x\right)\right\}$$

and $\{1, e^{ix}, \ldots, e^{i(m-1)x}\}$, respectively. Also, let U_m and U_m^* be the multiplication operators defined by

$$(U_m f)(x) = \exp\left(i\frac{m-1}{2}x\right)f(x)$$

and

$$(U_m^* f)(x) = \exp\left(-i\frac{m-1}{2}x\right)f(x).$$

We note that $U_m(S_m) = \tilde{S}_m$ and $U_m^*(\tilde{S}_m) = S_m$, and if $s(x) \in S_m$ then $s(x)\sin((x-c)/2)$, $s(x)\cos((x-c)/2) \in S_{m+1}$ for any real constant c.

Furthermore, if $s(x) \in S_m$ is real valued, then $s(x)$ can be written in the form (1.1) if m is odd and in the form (1.3) if m is even.

Also let L_m be the differential operators defined by $L_0 = I$ (the identity operator), $L_1 = D$ ($\equiv d/dx$), and for $m \geqslant 2$

$$L_m = \left(D^2 + \left(\frac{m-1}{2}\right)^2\right) L_{m-2}. \tag{2.1}$$

We observe that $S_m = \ker(L_m)$ for $m \geqslant 1$.

Correspondingly, we let M_m be the differential operators

$$M_m = D(D-i)\cdots(D-(m-1)i) \qquad \text{for} \quad m \geqslant 1.$$

Then $\tilde{S}_m = \ker(M_m)$ and

$$L_m = U_m^* M_m U_m \qquad \text{for} \quad m \geqslant 1. \tag{2.2}$$

For $m \geqslant 0$ now let $x_0 < x_1 < \cdots < x_m$ be distinct points in $\mathbb{R}$ such that

$$x_m - x_0 < 2\pi. \tag{2.3}$$

If $f \in C([x_0, x_m])$ we define a trigonometric analog of classical polynomial divided differences by

$$[x_0, x_1, \ldots, x_m]_t f \tag{2.4}$$
$$= 2^{m-1} \frac{\det\begin{pmatrix} 1 & \cos x & \sin x & \cdots & \cos nx & \sin nx & f(x) \\ x_0 & x_1 & x_2 & \cdots & x_{m-2} & x_{m-1} & x_m \end{pmatrix}}{\det\begin{pmatrix} \cos\frac{x}{2} & \sin\frac{x}{2} & \cdots & \cos\left(n+\frac{1}{2}\right)x & \sin\left(n+\frac{1}{2}\right)x \\ x_0 & x_1 & \cdots & x_{m-1} & x_m \end{pmatrix}}$$

if $m = 2n+1$, and if $m = 2n$

$$[x_0, x_1, \ldots, x_m]_t f \tag{2.5}$$
$$= 2^m \frac{\det\begin{pmatrix} \cos\frac{x}{2} & \sin\frac{x}{2} & \cdots & \cos\left(n-\frac{1}{2}\right)x & \sin\left(n-\frac{1}{2}\right)x & f(x) \\ x_0 & x_1 & & x_{m-2} & x_{m-1} & x_m \end{pmatrix}}{\det\begin{pmatrix} 1 & \cos x & \sin x & \cdots & \cos nx & \sin nx \\ x_0 & x_1 & x_2 & \cdots & x_{m-1} & x_m \end{pmatrix}}.$$

Here we have used the abbreviation

$$\det\begin{pmatrix}\varphi_0 & \varphi_1 & \cdots & \varphi_m\\ x_0 & x_1 & \cdots & x_m\end{pmatrix}$$

for the determinant

$$\begin{vmatrix}\varphi_0(x_0) & \varphi_1(x_0) & \cdots & \varphi_m(x_0)\\ \vdots & \vdots & & \vdots\\ \varphi_0(x_m) & \varphi_1(x_m) & \cdots & \varphi_m(x_m)\end{vmatrix}.$$

We note that since the system

$$\left\{\cos\frac{m-k}{2}x, \sin\frac{m-k}{2}x\right\}, \qquad k = 1, 3, 5,..., k \leqslant m$$

(with $\sin 0\, x \equiv 0$ excluded if m is odd), is a Chebyshev system on $[0\ 2\pi)$, $[x_0, x_1,..., x_m]_t f$ is well defined. If f is sufficiently differentiable, then as usual the definition of $[x_0, x_1,..., x_m]_t f$ is extended by continuity to the case when some of the x_j's are equal. Hence, if $y_1, y_2,..., y_r$ are the distinct members among $x_0, x_1,..., x_m$ such that precisely μ_k of the x_j's are equal to y_k, then there are constants $c_{k,\nu}$ such that

$$[x_0, x_1,..., x_m]_t f = \sum_{k=1}^{r}\sum_{\nu=1}^{\mu_k} c_{k,\nu} f^{(\nu-1)}(y_k). \tag{2.6}$$

Note also that if $f \in S_m$, then $[x_0, x_1,..., x_m]_t f = 0$.

In order to analyze these trigonometric divided differences, we also introduce an exponential version of divided differences. If $x_0 < x_1 < \cdots < x_m$ satisfies (2.3), then we define

$$\langle x_0, x_1,..., x_m\rangle f = \frac{\det\begin{pmatrix}1 & e^{ix} & \cdots & e^{i(m-1)x} & f(x)\\ x_0 & x_1 & \cdots & x_{m-1} & x_m\end{pmatrix}}{\det\begin{pmatrix}1 & e^{ix} & \cdots & e^{imx}\\ x_0 & x_1 & \cdots & x_m\end{pmatrix}}.$$

We note that if $f \in \tilde{S}_m$ then $\langle x_0, x_1,..., x_m\rangle f = 0$ and if $p(x) \equiv \sum_{k=0}^{m} a_k e^{ikx} \in \tilde{S}_{m+1}$ is the unique function in $\tilde{S}_{m+1}$ such that $p(x_j) = f(x_j)$, $j = 0, 1,..., m$, then $\langle x_0, x_1,..., x_m\rangle f = a_m$.

If some x_j's are equal, then $\langle x_0, x_1,..., x_m\rangle f$ is defined by continuity as above.

If we let φ be the mapping from $[x_0, x_0 + 2\pi)$ onto the unit circle in $\mathbb{C}$ defined by $\varphi(x) = e^{ix}$, then

$$\langle x_0, x_1,..., x_m\rangle f = [\varphi(x_0), \varphi(x_1),..., \varphi(x_m)] f \circ \varphi^{-1}, \tag{2.7}$$

where $[x_0, x_1, \ldots, x_m]f$ denotes the classical polynomial divided difference. Hence most known results for $[x_0, x_1, \ldots, x_m]f$ can be transferred to the exponential case. Particularly we have the difference formula

$$\langle x_0, x_1, \ldots, x_m\rangle f = \frac{\langle x_1, \ldots, x_m\rangle f - \langle x_0, \ldots, x_{m-1}\rangle f}{e^{ix_m} - e^{ix_0}}, \tag{2.8}$$

and the Leibniz rule

$$\langle x_0, x_1, \ldots, x_m\rangle f \cdot g = \sum_{k=0}^{m} \langle x_0, \ldots, x_k\rangle f \langle x_k, \ldots, x_m\rangle g. \tag{2.9}$$

Also, if all the x_j's are distinct then

$$\langle x_0, x_1, \ldots, x_m\rangle f = \sum_{j=0}^{m} \frac{f(x_j)}{\prod_{k\neq j} (e^{ix_j} - e^{ix_k})}. \tag{2.10}$$

The following lemma, which can be considered as a discrete version of the identity (2.2), describes the relation between the trigonometric and the exponential divided differences.

LEMMA 2.1. *Assume that* $x_0 \leqslant x_1 \leqslant \cdots \leqslant x_m < x_0 + 2\pi$. *Then, if f is sufficiently smooth,*

$$[x_0, x_1, \ldots, x_m]_t f = c_{0,m}\langle x_0, x_1, \ldots, x_m\rangle U_m f, \tag{2.11}$$

where

$$c_{0,m} = (2i)^m \exp\left(\frac{i}{2} \sum_{k=0}^{m} x_k\right).$$

Proof. Since both the right- and left-hand sides of (2.11) depend continuously on $x_0, x_1, \ldots, x_m$, we can assume that $x_0 < x_1 < \cdots < x_m$. By the definitions (2.4) and (2.5) of $[x_0, x_1, \ldots, x_m]_t f$, by expressing sin and cos in exponential forms, by forming linear combinations of successive pairs of columns, and by rearranging columns we obtain

$$[x_0, x_1, \ldots, x_m]_t f$$

$$= (2i)^m \frac{\det\begin{pmatrix} \exp\left(-i\frac{m-1}{2}x\right) & \exp\left(-i\frac{m-3}{2}x\right) & \cdots & \exp\left(i\frac{m-1}{2}x\right) & f(x) \\ x_0 & x_1 & \cdots & x_{m-1} & x_m \end{pmatrix}}{\det\begin{pmatrix} \exp\left(-i\frac{m}{2}x\right) & \exp\left(-i\frac{m-2}{2}x\right) & \cdots & \exp\left(i\frac{m}{2}x\right) \\ x_0 & x_1 & \cdots & x_m \end{pmatrix}},$$

or

$$[x_0, x_1, \ldots, x_m]_t f$$
$$= (2i)^m \exp\left(\frac{i}{2} \sum_{k=0}^{m} x_k\right) \langle x_0, x_1, \ldots, x_m \rangle \exp\left(i \frac{m-1}{2} x\right) f(x).$$

The next result is a trigonometric analog of a well-known representation of polynomial divided differences when the points x_j are distinct.

LEMMA 2.2. *Assume that $x_0 < x_1 < \cdots < x_m < x_0 + 2\pi$. Then for any $f \in C([x_0, x_m])$*

$$[x_0, x_1, \ldots, x_m]_t f = \sum_{j=0}^{m} \frac{f(x_j)}{\prod_{k \neq j} \sin((x_j - x_k)/2)}.$$

Proof. By (2.10) we obtain

$$\langle x_0, x_1, \ldots, x_m \rangle U_m f = \sum_{j=0}^{m} \frac{\exp(i((m-1)/2) x_j) f(x_j)}{\prod_{k \neq j} (e^{ix_j} - e^{ix_k})}.$$

The result now follows from the identity

$$e^{ix_j} - e^{ix_k} = 2i \exp\left(\frac{i}{2}(x_j + x_k)\right) \sin \frac{x_j - x_k}{2} \tag{2.12}$$

and Lemma 2.1.

Finally, we shall give a recurrence relation for the trigonometric divided differences. Since these differences are discrete analogs of the differential operators L_m, one would expect a simple relation between the differences of order m and $m - 2$.

LEMMA 2.3. *Assume that $m \geqslant 2$ and that $x_0 \leqslant x_1 \leqslant \cdots \leqslant x_m < x_0 + 2\pi$ is such that $x_0 < x_{m-1}$, $x_1 < x_m$. Then*

$$[x_0, x_1, \ldots, x_m]_t f = \gamma_{0,m}[x_2, \ldots, x_m]_t f + \beta_{0,m}[x_1, \ldots, x_{m-1}]_t f + \alpha_{0,m}[x_0, \ldots, x_{m-2}]_t f,$$

where

$$\begin{aligned}
\gamma_{j,m} &= 1\Big/\left[\left(\sin \frac{x_{j+m} - x_j}{2}\right)\left(\sin \frac{x_{j+m} - x_{j+1}}{2}\right)\right], \\
\beta_{j,m} &= -\left(\sin \frac{x_{j+m} + x_{j+m-1} - x_{j+1} - x_j}{2}\right)\Big/ \\
&\quad \left[\left(\sin \frac{x_{j+m} - x_j}{2}\right)\left(\sin \frac{x_{j+m} - x_{j+1}}{2}\right)\left(\sin \frac{x_{j+m-1} - x_j}{2}\right)\right], \\
\alpha_{j,m} &= 1\Big/\left[\left(\sin \frac{x_{j+m} - x_j}{2}\right)\left(\sin \frac{x_{j+m-1} - x_j}{2}\right)\right].
\end{aligned} \tag{2.13}$$

Proof. Let

$$c_{j,m} = (2i)^m \exp\left(\frac{i}{2} \sum_{k=0}^{m} x_{k+j}\right).$$

From Lemma 2.1 we then have

$$[x_0, x_1, \ldots, x_m]_t f = c_{0,m} \langle x_0, x_1, \ldots, x_m \rangle U_m f,$$

and since $(U_m f)(x) = e^{ix}(U_{m-2} f)(x)$ it follows from (2.8) and (2.9) that

$$\begin{aligned}(e^{ix_m} - e^{ix_0}) &\langle x_0, x_1, \ldots, x_m \rangle U_m f \\ &= \langle x_1, \ldots, x_m \rangle U_m f - \langle x_0, \ldots, x_{m-1} \rangle U_m f \\ &= e^{ix_1} \langle x_1, \ldots, x_m \rangle U_{m-2} f + \langle x_2, \ldots, x_m \rangle U_{m-2} f \\ &\quad - e^{ix_{m-1}} \langle x_0, \ldots, x_{m-1} \rangle U_{m-2} f - \langle x_0, \ldots, x_{m-2} \rangle U_{m-2} f.\end{aligned}$$

The desired result now follows by using (2.8) on the $(m-1)$st-order differences and by (2.12).

3. Trigonometric *B*-Splines

Suppose that $\{x_j\}$ is a nondecreasing sequence of real numbers satisfying $x_{j+m} - x_j < 2\pi$ for all j, where $m \geqslant 1$ is a given integer. We define real-valued functions $T_{j,m}$ on $\mathbb{R}$ by $T_{j,m}(x) \equiv 0$ if $x_{j+m} = x_j$ and

$$T_{j,m}(x) = [x_j, x_{j+1}, \ldots, x_{j+m}]_t \left(\sin \frac{y - x}{2}\right)_+^{m-1} \tag{3.1}$$

if $x_{j+m} > x_j$. Here the divided difference is taken with respect to y, and 0^0 is defined to be 0. As usual

$$\begin{aligned}(\sin z)_+ &= \sin z \qquad \text{if} \quad z > 0, \\ &= 0 \qquad \text{otherwise.}\end{aligned}$$

The functions $T_{j,m}$ are right continuous and

$$\begin{aligned}T_{j,1}(x) &= 1 \Big/ \sin \frac{x_{j+1} - x_j}{2} \qquad \text{if} \quad x_j \leqslant x < x_{j+1}, \\ &= 0 \qquad \text{otherwise.}\end{aligned} \tag{3.2}$$

We note that if $m = 2n + 1$ is odd, then the functions $T_{j,m}$ are essentially the trigonometric B-splines defined in [5]. These functions were defined as

$$c_m \frac{\det\begin{pmatrix} 1 & \cos y & \sin y & \cdots & \cos ny & \sin ny & (\sin((y-x)/2))_+^{m-1} \\ x_0 & x_1 & x_2 & \cdots & x_{m-2} & x_{m-1} & x_m \end{pmatrix}}{\det\begin{pmatrix} 1 & \cos y & \sin y & \cdots & \cos ny & \sin ny & y \\ x_0 & x_1 & x_2 & \cdots & x_{m-2} & x_{m-1} & x_m \end{pmatrix}},$$

where $c_m = \pi\, 2^{m-1}(n!)^2/(m-1)!$.

Similar to (3.1), we also define complex-valued functions $E_{j,m}$ on $\mathbb{R}$ by $E_{j,m}(x) \equiv 0$ if $x_{j+m} = x_j$ and

$$E_{j,m}(x) = \langle x_0, x_1, \ldots, x_m \rangle (e^{iy} - e^{ix})_+^{m-1}, \tag{3.3}$$

where

$$\begin{aligned} (e^{iy} - e^{iy})_+ &= e^{iy} - e^{ix} \qquad \text{if} \quad y > x, \\ &= 0 \qquad\qquad\quad \text{otherwise.} \end{aligned}$$

As in the polynomial case (see [1]) it now follows from (2.8) and (2.9) that if $x_{j+m} > x_j$ then $E_{j,m}$ satisfies the recurrence relation

$$E_{j,m}(x) = \frac{(e^{ix} - e^{ix_j})\, E_{j,m-1}(x) + (e^{ix_{j+m}} - e^{ix})\, E_{j+1,m-1}(x)}{e^{x_{j+m}} - e^{x_j}}. \tag{3.4}$$

Also, by differentiating (3.3) and by (2.8) we obtain

$$E'_{j,m}(x) = i(m-1)\, e^{ix} \frac{E_{j,m-1}(x) - E_{j+1,m-1}(x)}{e^{ix_{j+m}} - e^{ix_j}}. \tag{3.5}$$

From (2.12) and Lemma 2.1 we obtain the relation

$$T_{j,m}(x) = 2iU_m^* E_{j,m}(x) \exp\left(\frac{i}{2} \sum_{k=0}^{m} x_{j+k}\right). \tag{3.6}$$

Hence most properties of the functions $T_{j,m}$ can be derived from similar properties of the functions $E_{j,m}$.

THEOREM 3.1. *Suppose that* $m \geqslant 2$ *and* $x_j < x_{j+m} < x_j + 2\pi$. *Then* $T_{j,m}$ *satisfies the relation* (1.4).

Proof. By (2.12), (3.4), and (3.6) we obtain

$$\begin{aligned} \left(\sin \frac{x_{j+m} - x_j}{2}\right) T_{j,m}(x) = \exp\left(\frac{i}{2} \sum_{k=1}^{m-1} (x_{j+k} - x)\right) ((e^{ix} - e^{ix_j})\, E_{j,m-1}(x) \\ + (e^{ix_{j+m}} - e^{ix})\, E_{j+1,m-1}(x)), \end{aligned}$$

and hence (1.4) follows from another application of (2.12) and (3.6).

We note that if $t(x) = \sum_j c_j T_{j,m}(x)$ then by (1.4) the value $t(x)$ can be computed by forming positive linear combinations of the coefficients c_j. Such an algorithm was derived in [1] in the polynomial case. It also follows immediately from (1.4) and (3.2) that

$$T_{j,m}(x) > 0 \qquad \text{for} \quad x \in (x_j, x_{j+m}).$$

A similar result was obtained by Karlin [3, p. 524] for Chebyshevian splines using total positivity arguments. Note also that

$$T_{j,m}(x) = 0 \qquad \text{if} \quad x \notin [x_j, x_{j+m}).$$

This can of course also be seen directly from (3.1).

Similar to the proof of Theorem 3.1, the following differentiation formula for the functions $T_{j,m}$ follows from (3.4) and (3.5).

THEOREM 3.2. *Suppose that $m \geqslant 2$ and $x_j < x_{j+m} < x_j + 2\pi$. Then*

$$T'_{j,m}(x) = \left(\frac{m-1}{2}\right) \frac{\left(\cos \frac{x - x_j}{2}\right) T_{j,m-1}(x) - \left(\cos \frac{x_{j+m} - x}{2}\right) T_{j+1,m-1}(x)}{\sin \frac{x_{j+m} - x_j}{2}}. \tag{3.7}$$

Note the similarity of (3.7) to the formula

$$Q'_{j,m}(x) = (m-1) \frac{Q_{j,m-1}(x) - Q_{j+1,m-1}(x)}{x_{j+m} - x_j}$$

for differentiating polynomial B-splines. However, since the coefficients of $T_{k,m-1}$ in (3.7) depend on x, it will be more complicated to take higher derivatives in the trigonometric case.

Let D_m be the differential operatoı $D_m = D^2 + ((m-1)/2)^2$. The following formula, which is valid if $x_j < x_{j+m-1}$, $x_{j+1} < x_{j+m}$ and $x_{j+m} - x_j < 2\pi$, can be derived from Lemma 2.3 or by differentiating both sides of (3.7)

$$\begin{aligned} D_m T_{j,m}(x) = \frac{(m-1)(m-2)}{4} (&\alpha_{j,m} T_{j,m-2}(x) \\ &+ \beta_{j,m} T_{j+1,m-2}(x) + \gamma_{j,m} T_{j+2,m-2}(x)), \end{aligned} \tag{3.8}$$

where $\alpha_{j,m}$, $\beta_{j,m}$, and $\gamma_{j,m}$ are given by (2.13).

Let $t(x) = \sum_j c_j T_{j,m}(x)$ and suppose that $x_{r+1} > x_r$. If we assume that $m = 2n + 1$ then $t(x)$ also has the representation

$$t(x) = a_0 + \sum_{k=1}^{n} (a_k \cos k(x - \hat{x}) + b_k \sin k(x - \hat{x})) \tag{3.9}$$

on $[x_r, x_{r+1})$, where $\hat{x} \in [x_r, x_{r+1})$ is given. By using (3.7) and (3.8) the coefficients a_k and b_k can easily be computed. In order to see this, define differential operators V_k and W_k by $V_0 = I$ and for $k = 1, 2, \ldots, m$

$$V_k = D_{m-2k+2} V_{k-1}, \qquad W_k = D V_{k-1}.$$

By (3.8) we obtain

$$V_k t(x) = \sum_j c_j^{(k)} T_{j,m-2k}(x) \qquad \text{for} \quad k = 0, 1, \ldots, n,$$

where $c_j^{(0)} = c_j$, and for $k = 0, 1, \ldots, n-1$

$$c_j^{(k+1)} = \frac{(r-1)(r-2)}{4} (\alpha_{j,r} c_j^{(k)} + \beta_{j-1,r} c_{j-1}^{(k)} + \gamma_{j-2,r} c_{j-2}^{(k)}),$$

where $r = m - 2k$.

Similarly by (3.7)

$$W_k t(x) = \sum_j d_j^{(k)}(x)\, T_{j,m-2k+1}(x) \qquad \text{for} \quad k = 1, 2, \ldots, n,$$

where

$$d_j^{(k)}(x) = \left(\frac{r-1}{2}\right) \left(\frac{c_j^{(k-1)} \cos \dfrac{x - x_j}{2}}{\sin \dfrac{x_{j+r} - x_j}{2}} - \frac{c_{j-1}^{(k-1)} \cos \dfrac{x_{j+r-1} - x}{2}}{\sin \dfrac{x_{j+r-1} - x_{j-1}}{2}} \right)$$

and $r = m - 2k$.

Thus, at any point x, $V_k t(x)$ and $W_k t(x)$ can be found by taking weighted differences of the coefficients c_j and by using (1.4). If we now successively apply $V_0, V_1, \ldots, V_n$ to (3.9) at $x = \hat{x}$ we find that the coefficients a_k satisfy a triangular linear system with the solution given by

$$a_k = \left(V_{n-k} t(\hat{x}) - \sum_{j=0}^{k-1} \theta_{j,n-k} a_j \right) \Big/ \theta_{k,n-k}, \qquad k = 0, 1, \ldots, n,$$

where $\theta_{k,0} = 1$ and for $r \geqslant 1$

$$\theta_{k,r} = \prod_{j=0}^{r-1} ((n-j)^2 - k^2), \qquad 0 \leqslant k \leqslant n - r.$$

Similarly by applying $W_1, W_2, \ldots, W_n$ to (3.9) at $x = \hat{x}$ we obtain

$$b_k = \left(W_{n-k+1} t(\hat{x}) - \sum_{j=1}^{k-1} j \theta_{j,n-k} b_j \right) \Big/ (k \theta_{k,n-k}), \qquad k = 1, 2, \ldots, n.$$

Note that if m is even, a piecewise representation of $t(x)$ can be derived by an algorithm similar to the one described above.

We next give a trigonometric analog of an integral representation of divided differences. If $x_0 < x_m < x_0 + 2\pi$ we have for f sufficiently smooth

$$[x_0, x_1, ..., x_m]_t f = \frac{2^{m-1}}{(m-1)!} \int_{x_0}^{x_m} T_{0,m}(y) L_m f(y)\, dy,$$

where L_m is given by (2.1). This can be proved either via a similar identity involving $\langle x_0, x_1, ..., x_m \rangle$ and the function $E_{0,m}$ or by the following trigonometric Taylor expansion:

$$f(x) = s_m(x) + \frac{2^{m-1}}{(m-1)!} \int_{x_0}^{x} \left(\sin \frac{x-y}{2}\right)^{m-1} L_m f(y)\, dy. \tag{3.10}$$

Here $s_m \in S_m$ is determined by $s_0(x) \equiv 0$, $s_1(x) = f(x_0)$ and for $k \geqslant 0$

$$s_{k+2}(x) = s_k(x) + \frac{2^k}{k!} \left(\sin \frac{x-x_0}{2}\right)^k \cos \frac{x-x_0}{2} L_k f(x_0)$$
$$+ \frac{2^{k+1}}{(k+1)!} \left(\sin \frac{x-x_0}{2}\right)^{k+1} L_k f'(x_0).$$

Formula (3.10) follows by induction and an integration-by-parts argument.

For completeness, we also give a trigonometric analog for Marsden's identity. Let $\hat{E}_{j,m}(x) = (e^{ix_{j+m}} - e^{ix_j}) E_{j,m}(x)$ and $\hat{T}_{j,m}(x) = (\sin((x_{j+m} - x_j)/2)) \times T_{j,m}(x)$. By (3.6) we then have

$$\hat{T}_{j,m}(x) = U_m^* \hat{E}_{j,m}(x) \exp\left(i \frac{m-1}{2} \xi_{j,m}\right), \tag{3.11}$$

where $\xi_{j,m} = (x_{j+1} + \cdots + x_{j+m-1})/(m-1)$.

Now let I be any nonempty interval of the form (x_k, x_r) such that at least m x_j values are $\leqslant x_k$ and at least m x_j values are $\geqslant x_r$. As in the polynomial case (see [1, 4]) it now follows that

$$(e^{iy} - e^{ix})^{m-1} = \sum_j \hat{E}_{j,m}(x) \prod_{k=1}^{m-1} (e^{iy} - e^{ix_{j+k}}) \tag{3.12}$$

for $x \in I$ and $y \in \mathbb{R}$. By (2.12) and (3.11) we therefore obtain

$$\left(\sin \frac{y-x}{2}\right)^{m-1} = \sum_j \hat{T}_{j,m}(x) \prod_{k=1}^{m-1} \sin\left(\frac{y-x_{j+k}}{2}\right).$$

By expanding both sides of (3.12) in powers of e^{iy} and by comparing coefficients, we obtain for $x \in I$

$$e^{ikx} = \binom{m-1}{k}^{-1} \sum_j \sigma_{j,k}^{(m)} \hat{E}_{j,m}(x), \qquad k = 0, 1,\ldots, m-1,$$

where $\sigma_{j,k}^{(m)}$ are defined by

$$\sum_{k=0}^{m-1} (-1)^k \, \sigma_{j,k}^{(m)} e^{i(m-1-k)x} = \sum_{k=1}^{m-1} (e^{ix} - e^{ix_{j+k}}).$$

Now let

$$\tau_{j,k}^{(m)} = \binom{m-1}{k+(m-1)/2}^{-1} \sigma_{j,k+(m-1)/2}^{(m)} \exp\left(-i \frac{m-1}{2} \xi_{j,m}\right)$$

for $k = -(m-1)/2, -(m-1)/2+1,\ldots, (m-1)/2$. Then it follows from (3.11) that for $x \in I$ and $k \in \{-(m-1)/2, -(m-1)/2+1,\ldots, (m-1)/2\}$

$$e^{ikx} = \sum_j \tau_{j,k}^{(m)} \hat{T}_{j,m}(x), \tag{3.13}$$

and hence, by taking real and imaginary parts, we can express $\cos kx$ and $\sin kx$ as real linear combinations of $\hat{T}_{j,m}(x)$.

Suppose that $x_q < x_{q+1}$. If m is odd then it follows from [5] that the functions $\{\hat{T}_{q+1-m,m},\ldots, \hat{T}_{q,m}\}$ are linearly independent on (x_q, x_{q+1}). We note that this linear independence also follows from (3.13) for any $m \geqslant 1$. For since the functions $\hat{T}_{q+1-m,m},\ldots, \hat{T}_{q,m}$ span an m-dimensional linear space of functions on (x_q, x_{q+1}), they must be linearly independent. Hence we can show that the trigonometric B-splines $T_{j,m}$ span a space of trigonometric splines. To this end suppose $z_0 < z_1 < \cdots < z_{J+1}$, where $z_{j+1} - z_j < 2\pi$, and let $r_1, r_2,\ldots, r_J$ be integers such that $1 \leqslant r_j \leqslant m$. Now define $\Upsilon_m(z, r)$ to be the set of all real-valued function t defined on $[z_0, z_{J+1})$ such that

(i) $t|_{[z_j, z_{j+1})} \in S_m$, $\qquad j = 0, 1,\ldots, J,$

(ii) $t^{(k)}(z_j-) = t^{(k)}(z_j+)$, $\qquad j = 1, 2,\ldots, J,\ 0 \leqslant k \leqslant m-1-r_j$.

The elements of $\Upsilon_m(z, r)$ are called trigonometric splines (with interior knots $z_1, z_2,\ldots, z_J$ with multiplicities $r_1, r_2,\ldots, r_J$). In the case when m is odd it was essentially shown in [5] that every $t \in \Upsilon_m$ can be uniquely written in the form

$$t(x) = s_m(x) + \sum_{j=1}^{J} \sum_{k=1}^{r_j} c_{j,k} \frac{\partial^{k-1}}{\partial z_j^{k-1}} \left(\sin \frac{z_j - x}{2}\right)_+^{m-1},$$

where $s_m \in S_m$. This of course also holds when m is even. Thus

$$\dim Y_m = N + m, \qquad \text{where} \qquad N = \sum_{j=1}^{J} r_j .$$

As in the polynomial case we can now define a nondecreasing sequence $\{x_j\}_{j=1-m}^{N+m}$ such that $\{x_1, x_2, \ldots, x_N\} = \{z_1, z_2, \ldots, z_J\}$ and where r_j of the x_r's are equal to z_j. Now the functions $\{T_{j,m}\}_{j=1-m}^{N}$ form a basis for the space Y_m.

4. Remarks

1. Schoenberg [5] considers the subspace $\mathring{Y}_m$ of Y_m which consists of all periodic trigonometric splines, i.e.,

$$\mathring{Y}_m(z, r) = \{t \in Y_m(z, r) \mid t^{(k)}(z_0+) = t^{(k)}(z_{J+1}-),\ k = 0, 1, \ldots, m-1\}.$$

The space $\mathring{Y}_m$ has dimension N and a local basis can be constructed from the functions $\{T_{j,m}\}$ by techniques analogous to the one used for the example in [7] in the polynomial case.

2. In addition to the recurrence relations (1.2), (1.4) and (3.4), we also mention the relation

$$C_{j,m}(x) = \frac{(\cos x - \cos x_j)\, C_{j,m-1}(x) + (\cos x_{j+m} - \cos x)\, C_{j+1,m-1}(x)}{\cos x_{j+m} - \cos x_j} \tag{3.14}$$

with

$$\begin{aligned} C_{j,1}(x) &= 1/(\cos x_{j+1} - \cos x_j) && x_j \leqslant x < x_{j+1} \\ &= 0 && \text{otherwise.} \end{aligned}$$

Here we assume that $x_j < x_{j+m} < x_j + \pi$.

The functions $C_{j,m}$ are piecewise of the form

$$\sum_{k=0}^{m-1} a_k \cos kx.$$

To derive (3.14) we simply take $\varphi(x) = \cos x$ in (2.7) and proceed as in the derivation of (3.4).

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